Pre-class Warm-up!!!

If we have a function f : [a,b] -> R , what does the Fundamental Theorem of Calculus say?

a. $\int_a^b f\,dx = \frac{df}{dx}\Big|_b - \frac{df}{dx}\Big|_a$

✓b. $\int_a^b \frac{df}{dx} = f(b) - f(a)$

c. $\int_a^b \frac{df}{dx} = f(x) + (b-a)$

d. $\frac{d}{dx}\int_a^b f\,dx = f(x)$

e. None of the above

FAQs

1. What is the quiz on?

Answr: the material we did last week

2. What resources do I have to study?

Answer. Go to the module for this week

You see

- Practice quiz
- videos

In Homework and readings you find

- Carter's notes
- Webb's notes
- HW question list
- Insight readings.

If Webb's lectures aren't there yet, go to Webb's home page and click on lecture notes.

Section 8.1 Green's theorem

We learn:

- What the theorem says
- A couple of ways of writing it
- How it is useful to make calculation easier
- How it can be used to compute area

George Green (1793 - 1841) stated his theorem in 1828 in a pamphlet about electricity and magnitism.

Also, Mikhail Ostrogradsky (1801-1861) stated the theorem in 1828 as well.

Green's theorem.

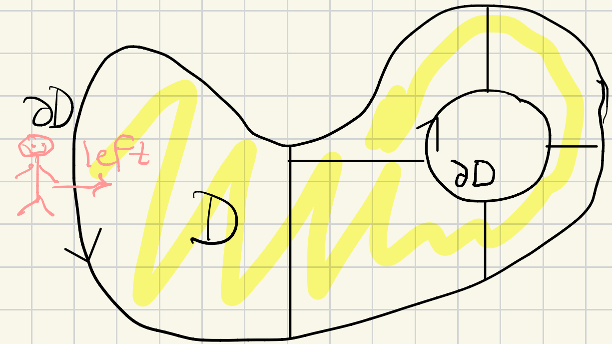
Let D be a bounded region in R^2 that is the union of simple regions with boundary a union of finitely many simple closed curves. Orient the boundary ∂D so that D is on the left.

If F(x,y) = (P(x,y) , Q(x,y)) is a vector field then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\int_{\partial D} F \cdot d\underline{s} \; = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

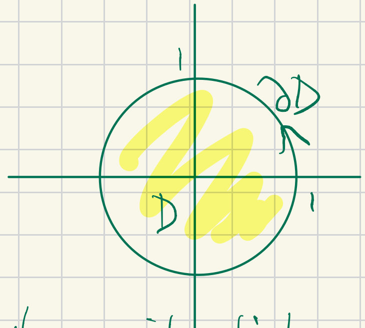
The left side can also be written and computes the work done by F in going round the boundary. The right side can have dA instead of dx dy.



In the book they write C^+ for the boundary of D oriented in this way.

It's a 2-D version of the fund. thm of calculus.

Example.
Let $F(x,y) = (e^{\sin x} + 2xy, \sqrt{y^3+2} + x^2 + x)$.
Find the work done by F in moving counterclockwise round the unit circle $x^2 + y^2 = 1$.



Solution Let D be the unit disk.

$$\int_{\partial D} F \cdot ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_D (2x + 1 - 2x)\, dx\,dy = \iint_D 1\, dx\,dy$$

$$= \text{Area of } D = \pi$$

B/c $\dfrac{\partial Q}{\partial x} = 0 + 2x + 1$

$\dfrac{\partial P}{\partial y} = 0 + 2x$

Example.

The following integrals compute the area of D :

$$\int_{\partial D} x\,dy \quad , \quad \int_{\partial D} -y\,dx \quad , \quad \frac{1}{2}\int_{\partial D} x\,dy - y\,dx .$$

$$\|$$

$$\iint_D \left(\frac{\partial x}{\partial x} - 0\right) dx\,dy = \int_D 1\,dx\,dy$$

Find the area of the ellipse parametrized by
c(t) = (2 cos t , 3 sin t).

Green's theorem $\displaystyle\int_{\partial D} P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Proof

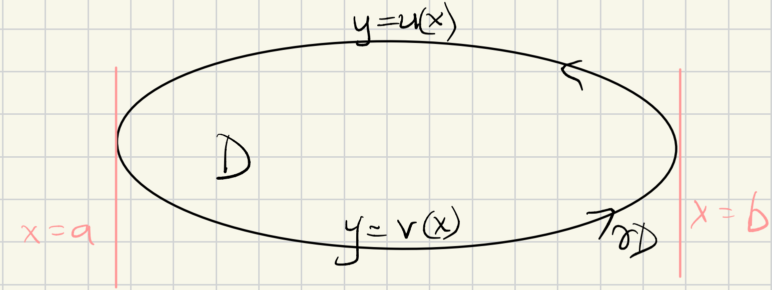
We show separately $\displaystyle\int_{\partial D} P\,dx = \int_D -\frac{\partial P}{\partial y}\,dx\,dy$

and $\displaystyle\int_{\partial D} Q\,dy = \int_D \frac{\partial Q}{\partial x}\,dx\,dy$

We do this on simple regions D.

For a y-simple region we show

$$\int_{\partial D} P\,dx = \int_D -\frac{\partial P}{\partial y}\,dx\,dy$$



$$\int_{\partial D} P\,dx = \int_{x=b}^{x=a} P(x,u(x))\,dx + \int_{x=a}^{x=b} P(x,v(x))\,dx$$

$$= \int_a^b \big(-P(x,u(x)) + P(x,v(x))\big)\,dx$$

$$= \int_a^b \left(-\int_{v(x)}^{u(x)} \frac{\partial P}{\partial y}\,dy\right) dx$$

$$= \int_D -\frac{\partial P}{\partial y}\,dx\,dy$$

Other ways to write

$$\int_{\partial D} P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

1. $$\int_{\partial D} -P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\,dy$$

and now $\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}$ is $\operatorname{div}(Q,P)$.

The left side has an interpretation in terms of a unit normal vector to ∂D, see the book.

2. From $F(x,y) = (P,Q)$ construct the vector field $(P,Q,0)$ on $\mathbb{R}^3$.

Now $\nabla \times (P,Q,0) = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

so $$\int_{\partial D} P\,dx + Q\,dy = \int_D \nabla \times (P,Q,0) \cdot \underline{k}\; dA$$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the 'scalar curl' of (P,Q).